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Asymptotic lattices and their integrable reductions: I. The Bianchi–Ernst and the Fubini–Ragazzi lattices

A Doliwa[1], **M Nieszporski**[2] **and P M Santini**[3]

[1] Instytut Fizyki Teoretycznej, Uniwersytet Warszawski, ul. Hoża 69, 00-681 Warszawa, Poland
[2] Instytut Fizyki Teoretycznej, Uniwersytet w Białymstoku, ul. Lipowa 41, 15-424 Białystok, Poland
[3] Dipartimento di Fisica, Università di Roma 'La Sapienza', Istituto Nazionale di Fisica Nucleare, Sezione di Roma, P-le Aldo Moro 2, I–00185 Roma, Italy

E-mail: doliwa@fuw.edu.pl, maciejun@fuw.edu.pl, maciejun@uwb.edu.pl and paolo.santini@roma1.infn.it

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Abstract
We review recent results on asymptotic lattices and their integrable reductions. We present the theory of general asymptotic lattices in $\mathbb{R}^3$ together with the corresponding theory of their Darboux-type transformations. Then we find a novel permutability theorem for Bianchi surfaces, which can be reinterpreted as a discrete version of the Bianchi–Ernst system and coincides with an equation recently introduced by Schief (Schief W K 2001 *Stud. Appl. Math.* **106** 85–137). Using the well known connection between the Bianchi and Ernst systems, we also propose the discrete analogue of the Ernst system. Finally, we present the theory of the discrete analogues of isothermally asymptotic (Fubini–Ragazzi) nets together with their transformations.

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1. Introduction

One of the best known examples of integrable geometries is provided by asymptotic nets on surfaces of constant curvature, which are described by the sine-Gordon equation [1]. It turns out that asymptotic nets on surfaces in $\mathbb{E}^3$ provide other classes of integrable geometries, for example, Bianchi surfaces [1, 19], affine spheres [29] and isothermally asymptotic nets (Fubini–Ragazzi nets) [14, 21, 25].

The discrete analogues of asymptotic nets (asymptotic lattices) were proposed a long time ago by Sauer [26]. He also considered the 'discrete pseudospherical surfaces' whose study was recently undertaken by Bobenko and Pinkal [2] from the point of view of integrable systems. Recently Bobenko and Schief introduced the discrete analogue of (indefinite) affine spheres [3, 4], which is described by the discrete analogue of the Tzitzeica equation.

The integrability aspects of generic asymptotic lattices were the subject of studies by Nieszporski [21] and Doliwa [8]. In particular, in the paper of Doliwa, the theory of asymptotic lattices and their transformations was considered as a part of the theory of quadrilateral lattices (the discrete analogues of conjugate nets); for information about the quadrilateral lattices, their transformations and reductions see [9–12, 17]. The integrable discrete analogue of the isothermally asymptotic nets (Fubini–Ragazzi nets), which includes the discrete affine spheres as a particular integrable subreduction, was introduced by Nieszporski in [21, 22]. In a recent paper Schief [28] introduced the so-called 'discrete Calapso system':

$$\boldsymbol{N}_{(12)} + \boldsymbol{N} = \frac{U(m_1) + V(m_2)}{(\boldsymbol{N}_{(1)} + \boldsymbol{N}_{(2)}) \cdot (\boldsymbol{N}_{(1)} + \boldsymbol{N}_{(2)})} (\boldsymbol{N}_{(1)} + \boldsymbol{N}_{(2)}) \tag{1}$$

where $\cdot$ denotes the scalar product

$$\boldsymbol{A} \cdot \boldsymbol{B} := A_0 B_0 + \epsilon (A_1 B_1 + A_2 B_2) \qquad \epsilon = \pm 1 \tag{2}$$

and $U(m_1)$ is a function of m_1 only and $V(m_2)$ is a function of m_2. This equation is an integrable constraint for the discrete Moutard equation and therefore it can be interpreted, via the discrete Lelieuvre formulae of Konopelchenko and Pinkall [16], as an integrable reduction of asymptotic lattices.

In this paper we present the theory of asymptotic lattices and their integrable reductions from a unified perspective. In addition to the results already known in the literature, we develop the following new aspects.

(1) We show that equation (1) is also the proper discrete analogue of the Bianchi system. The proof, classical in the soliton literature, consists in showing that equation (1) is not only the permutability diagram of the Calapso system [28], but also the permutability diagram of the Bianchi system. Using the well known connection between the Bianchi and Ernst systems (see [20] and references therein), we also derive a natural discretization of the (hyperbolic) Ernst system. We also obtain, in the framework of Bianchi's work [1], the Darboux–Bäcklund transformations of equation (1) in a form different from that presented in [28]; this new formulation allows one to construct, in principle, solutions through a sequence of linear steps only.

(2) We present the theory of transformations of the discrete Fubini–Ragazzi nets [22] which allows one to construct solutions through a sequence of linear steps only. Therefore we prove the integrability of the Fubini–Ragazzi lattices introduced in [21].

This paper is organized as follows. Section 2 is devoted to the general theory of asymptotic lattices and their transformations. In section 3 we present the discrete analogues of the Bianchi and Fubini–Ragazzi reductions of the asymptotic nets. In appendix A we collected some results from the theory of quadrilateral lattices which are used in this paper. In appendix B we present basic notions of the line geometry of Plücker while in appendix C the superposition of Moutard transformations is recalled.

We use the following notation: given a function f defined on the two-dimensional integer lattice $\mathbb{Z}^2 \ni (m_1, m_2)$, we denote by $f_{(\pm i)}$, $i = 1, 2$, the function f of the shifted arguments, i.e. $f_{(\pm 1)}(m_1, m_2) = f(m_1 \pm 1, m_2)$, $f_{(\pm 2)}(m_1, m_2) = f(m_1, m_2 \pm 1)$ and $f_{(12)}(m_1, m_2) = f(m_1 + 1, m_2 + 1)$. We also make use of the following difference operators: $\Delta_i f = f_{(i)} - f$.

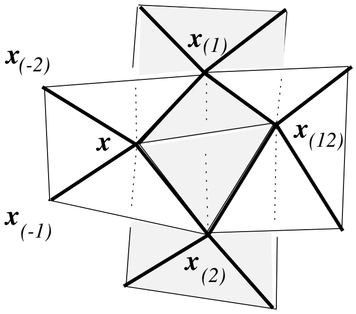

Figure 1. Asymptotic lattice.

2. Asymptotic lattices and W-congruences

In this section we present the theory of general asymptotic lattices. Of these lattices, characterized by linear difference equations (equations (3) or (16) below), there exist Darboux-type transformations whose superposition satisfies the permutability property. Therefore they can be coined *integrable lattices*.

2.1. Asymptotic lattices

The asymptotic lattice is defined as in the continuous case and, roughly speaking, it is a two-dimensional lattice such that osculating planes of the parametric curves coincide at the intersection point (see figure 1).

Definition 1 [26]. *An asymptotic lattice (or discrete asymptotic net) is a mapping* $\boldsymbol{x} : \mathbb{Z}^2 \to \mathbb{R}^3$ *such that any point* $\boldsymbol{x}$ *of the lattice is coplanar with its four nearest neighbours* $\boldsymbol{x}_{(1)}$, $\boldsymbol{x}_{(2)}$, $\boldsymbol{x}_{(-1)}$ *and* $\boldsymbol{x}_{(-2)}$.

Remark 1. The common plane of the five points $\boldsymbol{x}$, $\boldsymbol{x}_{(1)}$, $\boldsymbol{x}_{(2)}$, $\boldsymbol{x}_{(-1)}$ and $\boldsymbol{x}_{(-2)}$ is the tangent plane of the lattice at $\boldsymbol{x}$.

Remark 2. Throughout the paper we consider only non-degenerate asymptotic lattices, i.e. for every $\boldsymbol{x}$ the three vectors $\boldsymbol{x}_{(1)} - \boldsymbol{x}$, $\boldsymbol{x}_{(2)} - \boldsymbol{x}$ and $\boldsymbol{x}_{(12)} - \boldsymbol{x}$ are linearly independent.

Algebraically, the asymptotic lattice condition can be rewritten in the form of the following linear system [4, 21]:

$$\begin{aligned} \boldsymbol{x}_{(11)} - \boldsymbol{x}_{(1)} &= A(\boldsymbol{x}_{(1)} - \boldsymbol{x}) + P(\boldsymbol{x}_{(12)} - \boldsymbol{x}_{(1)}) \\ \boldsymbol{x}_{(22)} - \boldsymbol{x}_{(2)} &= B(\boldsymbol{x}_{(2)} - \boldsymbol{x}) + Q(\boldsymbol{x}_{(12)} - \boldsymbol{x}_{(2)}) \end{aligned} \tag{3}$$

which gives

$$\begin{aligned} \boldsymbol{x}_{(112)} - \boldsymbol{x}_{(12)} &= \frac{A_{(2)}}{H}(\boldsymbol{x}_{(12)} - \boldsymbol{x}_{(2)}) + \frac{P_{(2)}B_{(1)}}{H}(\boldsymbol{x}_{(12)} - \boldsymbol{x}_{(1)}) \\ \boldsymbol{x}_{(221)} - \boldsymbol{x}_{(12)} &= \frac{B_{(1)}}{H}(\boldsymbol{x}_{(12)} - \boldsymbol{x}_{(1)}) + \frac{Q_{(1)}A_{(2)}}{H}(\boldsymbol{x}_{(12)} - \boldsymbol{x}_{(2)}). \end{aligned} \tag{4}$$

The compatibility condition $\boldsymbol{x}_{(1122)} = \boldsymbol{x}_{(2211)}$ implies that the functions A, B, P, Q are constrained to [21]

$$\frac{A_{(22)}}{AH_{(2)}} = \frac{B_{(11)}}{BH_{(1)}} \tag{5}$$

$$\begin{aligned} \frac{A_{(22)}H}{A_{(2)}H_{(2)}}(1 + B - Q) &= D_{(1)} - Q_{(1)}C_{(2)} \\ \frac{B_{(11)}H}{B_{(1)}H_{(1)}}(1 + A - P) &= C_{(2)} - P_{(2)}D_{(1)} \end{aligned} \tag{6}$$

where the functions C, D, H are defined as

$$\begin{aligned} H &:= 1 - P_{(2)}Q_{(1)} \\ C &:= 1 + \frac{A_{(2)}}{H} + \frac{B_{(1)}P_{(2)}}{H} \\ D &:= 1 + \frac{B_{(1)}}{H} + \frac{A_{(2)}Q_{(1)}}{H}. \end{aligned} \tag{7}$$

Let us introduce [21] the discrete canonical tangent fields $\boldsymbol{W}$ and $\boldsymbol{Z}$ of the asymptotic lattice $\boldsymbol{x}$ by

$$\begin{aligned} \boldsymbol{x}_{(12)} - \boldsymbol{x}_{(2)} &= \alpha \boldsymbol{W} \\ \boldsymbol{x}_{(12)} - \boldsymbol{x}_{(1)} &= \beta \boldsymbol{Z} \end{aligned} \tag{8}$$

where functions α and β are defined by

$$\begin{aligned} \beta_{(2)} &= \frac{B_{(1)}}{H}\beta \\ \alpha_{(1)} &= \frac{A_{(2)}}{H}\alpha. \end{aligned} \tag{9}$$

Equations (4) take the form

$$\begin{aligned} \Delta_1 \boldsymbol{W} &= \mathcal{P}\boldsymbol{Z} \\ \Delta_2 \boldsymbol{Z} &= \mathcal{Q}\boldsymbol{W} \end{aligned} \tag{10}$$

in terms of fields $\boldsymbol{W}$ and $\boldsymbol{Z}$, where

$$\begin{aligned} \mathcal{P} &= \frac{P_{(2)}B_{(1)}}{A_{(2)}}\frac{\beta}{\alpha} \\ \mathcal{Q} &= \frac{Q_{(1)}A_{(2)}}{B_{(1)}}\frac{\alpha}{\beta}. \end{aligned} \tag{11}$$

Note that $H = 1 - \mathcal{P}\mathcal{Q}$.

Remark 3. The first-order system (10) appears, for example, as the linear problem of the two-dimensional quadrilateral lattice [9] (see also appendix A). We will use this fact in section 3.2 where we define the discrete analogue of the isothermally asymptotic (Fubini–Ragazzi) nets.

2.2. The discrete Moutard equation and the Lelieuvre representation of the asymptotic lattices

It can be shown [16, 21] that a suitable rescaled normal field $\boldsymbol{N}$ to generic asymptotic lattice $\boldsymbol{x}$ (for a more detailed discussion see [21]) is connected with the lattice itself by the discrete analogue of the Lelieuvre formulae

$$\begin{aligned} \Delta_1 \boldsymbol{x} &= \boldsymbol{N}_{(1)} \times \boldsymbol{N} \\ \Delta_2 \boldsymbol{x} &= \boldsymbol{N} \times \boldsymbol{N}_{(2)}. \end{aligned} \tag{12}$$

Moreover, there exists a function F such that the normal vector field $\boldsymbol{N}$ satisfies the discrete analogue of the Moutard equation [24, 27]

$$\boldsymbol{N}_{(12)} + \boldsymbol{N} = F(\boldsymbol{N}_{(1)} + \boldsymbol{N}_{(2)}). \tag{13}$$

Remark 4. There is an alternative version of the Lelieuvre type representation of asymptotic lattices and of the Moutard equation which differs from (12), (13) only by a change of signs:

$$\boldsymbol{N}_{(12)} - \boldsymbol{N} = F(\boldsymbol{N}_{(1)} - \boldsymbol{N}_{(2)}) \tag{14}$$

$$\begin{aligned} \Delta_1 \boldsymbol{x} &= \boldsymbol{N}_{(1)} \times \boldsymbol{N} \\ \Delta_2 \boldsymbol{x} &= \boldsymbol{N}_{(2)} \times \boldsymbol{N} \end{aligned} \tag{15}$$

see [21] for details. This minor modification becomes important when discussing the generation of additional dimensions of the lattice by the Darboux-type transformations.

Notice that, due to the Lelieuvre formulae (12), there exist functions γ and δ such that the normal $\boldsymbol{N}$ satisfies the linear system

$$\begin{aligned} \boldsymbol{N}_{(11)} - \boldsymbol{N}_{(1)} &= A(\boldsymbol{N}_{(1)} - \boldsymbol{N}) - P(\boldsymbol{N}_{(12)} - \boldsymbol{N}_{(1)}) + \gamma \boldsymbol{N}_{(1)} \\ \boldsymbol{N}_{(22)} - \boldsymbol{N}_{(2)} &= B(\boldsymbol{N}_{(2)} - \boldsymbol{N}) - Q(\boldsymbol{N}_{(12)} - \boldsymbol{N}_{(2)}) + \delta \boldsymbol{N}_{(2)}. \end{aligned} \tag{16}$$

The compatibility of equations (13)–(16) gives the relations between the functions F, γ, δ with the fields A, B, P, Q of the asymptotic lattice $\boldsymbol{x}$

$$FF_{(1)} = \frac{A_{(2)}}{AH} \qquad FF_{(2)} = \frac{B_{(1)}}{BH} \tag{17}$$

$$\begin{aligned} \gamma &= -1 - A - P + \frac{C}{F_{(1)}} \\ \delta &= -1 - B - Q + \frac{D}{F_{(2)}}. \end{aligned} \tag{18}$$

Note that the compatibility condition of equations (17) is provided by equation (5).

2.3. The discrete Moutard transformation

Given [24, 27] a scalar solution Θ of the Moutard equation (13)

$$\Theta_{(12)} + \Theta = F(\Theta_{(1)} + \Theta_{(2)}) \tag{19}$$

then the solution $\boldsymbol{N}'$ of the system of equations

$$\begin{aligned} (\boldsymbol{N}'_{(1)} \mp \boldsymbol{N}) &= \frac{\Theta}{\Theta_{(1)}}(\boldsymbol{N}' \mp \boldsymbol{N}_{(1)}) \\ (\boldsymbol{N}'_{(2)} \pm \boldsymbol{N}) &= \frac{\Theta}{\Theta_{(2)}}(\boldsymbol{N}' \pm \boldsymbol{N}_{(2)}) \end{aligned} \tag{20}$$

satisfies equation (13) with the transformed potential

$$F' = \frac{\Theta_{(1)}\Theta_{(2)}}{\Theta\ \Theta_{(12)}}F. \tag{21}$$

Remark 5. We consider [8, 21] two possibilities of signs in the Moutard transformation in order: (i) to preserve the symmetry between the variables m_1 and m_2, (ii) to interpret the transformation direction (denoted by a prime) as a shift in the third variable (see remark 4), and (iii) to reproduce the discrete Moutard equation in the superposition formula.

The algebraic superposition formula for two Moutard transformations is given in the following result [8, 21]:

Theorem 1. *Let $\boldsymbol{N}^{(1)}$ be the upper-sign Moutard transform of $\boldsymbol{N}$ with respect to Θ^1, $\boldsymbol{N}^{(2)}$ be the lower-sign Moutard transform of $\boldsymbol{N}$ with respect to Θ^2 and Ξ be the one parameter family of solutions of the system*

$$\begin{aligned} \Delta_1 \Xi &= \Theta^1_{(1)}\Theta^2 - \Theta^2_{(1)}\Theta^1 \\ \Delta_2 \Xi &= \Theta^2_{(2)}\Theta^1 - \Theta^1_{(2)}\Theta^2. \end{aligned} \tag{22}$$

Then the function $\boldsymbol{N}^{(12)}$, given by

$$\boldsymbol{N}^{(12)} + \boldsymbol{N} = \frac{\Theta^1\Theta^2}{\Xi}(\boldsymbol{N}^{(1)} + \boldsymbol{N}^{(2)}) \tag{23}$$

is simultaneously the lower-sign Moutard transform of $\boldsymbol{N}^{(1)}$ with respect to $\Theta^{2(1)} = \Xi/\Theta^1$ and the upper-sign Moutard transform of $\boldsymbol{N}^{(2)}$ with respect to $\Theta^{1(2)} = \Xi/\Theta^2$.

Remark 6. When interpreting the transformation shifts (upper indices in brackets) as shifts in discrete variables the formula (23) is of the form of the discrete Moutard equation.

2.4. The discrete W-congruences

It can be checked directly [8,21] that the lattice $\boldsymbol{x}'$ (we still use the $\pm$ convention of the Moutard transformation) defined by the formula

$$\boldsymbol{x}' = \boldsymbol{x} \pm \boldsymbol{N}' \times \boldsymbol{N} \tag{24}$$

is a new asymptotic lattice with the normal $\boldsymbol{N}'$ entering in its Lelieuvre representation.

Remark 7. Notice again the correspondence between the shifts generated by the Moutard transformation and the shifts in the discrete variables m_i, $i = 1, 2$. Namely, in the notation of theorem 1, the transformation formulae

$$\begin{aligned} \boldsymbol{x}^{(1)} - \boldsymbol{x} &= \boldsymbol{N}^{(1)} \times \boldsymbol{N} \\ \boldsymbol{x}^{(2)} - \boldsymbol{x} &= \boldsymbol{N} \times \boldsymbol{N}^{(2)} \end{aligned} \tag{25}$$

are of the form of the Lelieuvre representation.

The translation of $\boldsymbol{x}'$ by a constant vector is still an asymptotic lattice with the normal $\boldsymbol{N}'$. However, the lattice $\boldsymbol{x}'$ defined in (24) helps to define a certain family of lines called the discrete *W* (from Weingarten) congruences [8,21]. The family of straight lines connecting $\boldsymbol{x}$ and $\boldsymbol{x}'$ is at a tangent to both asymptotic lattices and has analogous properties to those of the *W* congruences known in the theory of transformations of asymptotic nets [1].

Definition 2 [8]. *By a discrete W-congruence we mean a two-parameter family of straight lines connecting two asymptotic lattices in such a way that the lines are tangent to both lattices in the corresponding points.*

Remark 8. It can be shown [8] that any discrete *W*-congruence can be constructed via the discrete Moutard transformation.

The permutability property of superpositions of the Moutard transformation implies the corresponding permutability of the transformations of the asymptotic lattices. The asymptotic lattice $\boldsymbol{x}^{(12)}$, corresponding to the superposition of the two Moutard transformations in theorem 1, is given by

$$\boldsymbol{x}^{(12)} = \boldsymbol{x} + \frac{\Theta^1\Theta^2}{\Xi}\boldsymbol{N}^{(1)} \times \boldsymbol{N}^{(2)}. \tag{26}$$

2.5. Discrete Jonas formulae

In this section we present [21] another useful description, introduced by Jonas [15] in the continuous case of the transformation of asymptotic lattices.

Let $\boldsymbol{N}'$ be a transform of $\boldsymbol{N}$ under the discrete Moutard transformation (we consider here the upper-sign transformation only). Define x^a, $a = 1, 2, 3$ as coefficients of the decomposition of $\Theta\boldsymbol{N}'$ in the basis $\{\boldsymbol{N}, \boldsymbol{N}_{(1)}, \boldsymbol{N}_{(2)}\}$

$$\Theta\boldsymbol{N}' = x^1\boldsymbol{N}_{(1)} + x^2\boldsymbol{N}_{(2)} + x^3\boldsymbol{N}. \tag{27}$$

After substitution of the above expression into the discrete Moutard transformation we obtain that the coefficients x^a satisfy six equations, which can be split into two parts: the following linear system for x^1 and x^2

$$\begin{aligned} x^1_{(2)} - Qx^2_{(2)} &= \frac{1}{F}x^1 \\ x^2_{(1)} - Px^1_{(1)} &= \frac{1}{F}x^2 \end{aligned} \tag{28}$$

and the remaining equations

$$\begin{aligned}
x^3 + \Theta_{(1)} &= -Ax^1_{(1)} - \frac{1}{F}x^2 \\
x^3 - \Theta_{(2)} &= -Bx^2_{(2)} - \frac{1}{F}x^1 \\
x^3_{(1)} + \Theta &= -(\gamma + 1 + A + P)x^1_{(1)} + x^1 - x^2 \\
x^3_{(2)} - \Theta &= -(\delta + 1 + B + Q)x^2_{(2)} + x^2 - x^1.
\end{aligned} \tag{29}$$

The new normal $\boldsymbol{N}'$ satisfies the primed analogue of equations (13) and (16), and the primed functions are related to the non-primed ones via

$$\begin{aligned}
P' &= \left(-P + \frac{S}{L}\frac{x^2}{F}\right)\frac{\Theta_{(12)}}{\Theta_{(11)}} \\
Q' &= \left(-Q + \frac{T}{L}\frac{x^1}{F}\right)\frac{\Theta_{(12)}}{\Theta_{(22)}} \\
A' &= A\left(1 + \frac{S}{L}x^1_{(1)}\right)\frac{\Theta}{\Theta_{(11)}} \\
B' &= B\left(1 + \frac{T}{L}x^2_{(2)}\right)\frac{\Theta}{\Theta_{(22)}}
\end{aligned} \tag{30}$$

in which

$$\begin{aligned}
S &:= \Theta_{(11)} - (\gamma + 1 + A + P)\Theta_{(1)} + A\Theta + P\Theta_{(12)} \\
T &:= \Theta_{(22)} - (\delta + 1 + B + Q)\Theta_{(2)} + B\Theta + Q\Theta_{(12)} \\
L &:= x^1\Theta_{(1)} + x^2\Theta_{(2)} + x^3\Theta.
\end{aligned} \tag{31}$$

The Jonas formulation gives an alternative way to construct transformations of asymptotic lattices.

Theorem 2. *Consider an asymptotic lattice $\boldsymbol{x}$ and its normal $\boldsymbol{N}$ connected by the Lelieuvre representation. Any non-zero solution (x^1, x^2) of the linear system (28) leads, via equations (29), to functions x^3 and Θ such that:*

(i) Θ satisfies the Moutard equation of $\boldsymbol{N}$;
(ii) $\boldsymbol{N}'$, given by (27), and $\boldsymbol{x}'$, given by the upper-sign version of (24), are the corresponding transforms of $\boldsymbol{N}$ and $\boldsymbol{x}$.

2.6. The Plücker geometry approach to asymptotic lattices and W congruences

In this section we present [8] the theory of asymptotic lattices and their transformations in the language of the line geometry of Plücker (see appendix B).

Denote by $\mathfrak{p}_i$, $i = 1, 2$ the bi-vectors representing the asymptotic lines of the lattice $\boldsymbol{x}$, i.e. the lines passing through points $\boldsymbol{x}$ and $\boldsymbol{x}_{(i)}$:

$$\mathfrak{p}_i = \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix} \wedge \begin{pmatrix} \boldsymbol{x}_{(i)} \\ 1 \end{pmatrix} \qquad i = 1, 2. \tag{32}$$

Equations (3) imply the following linear system:

$$\begin{aligned}
\mathfrak{p}_{1(1)} &= A\mathfrak{p}_1 + P\mathfrak{p}_{2(1)} \\
\mathfrak{p}_{2(2)} &= B\mathfrak{p}_2 + Q\mathfrak{p}_{1(2)}.
\end{aligned} \tag{33}$$

The planar pencils of straight lines are represented in the Plücker geometry by isotropic (i.e. contained in $\mathcal{Q}_P$) lines. Therefore the tangent planes of the asymptotic lattice are represented by a two-parameter family of isotropic lines. Since two neighbouring tangent planes, in $\boldsymbol{x}$

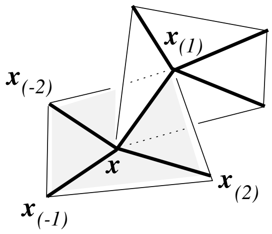

Figure 2. Asymptotic directions as focal lattices of the isotropic congruence.

and $\boldsymbol{x}_{(i)}$, intersect along the asymptotic line represented by $\mathfrak{p}_i$, then the corresponding two isotropic lines have one point in common (see figure 2). Using the terminology of the theory of quadrilateral lattices (see appendix A) the above considerations can be summarized as follows.

Theorem 3 [8]. *A discrete asymptotic net in $\mathbb{P}^3$, viewed as the envelope of its tangent planes, corresponds to a congruence of isotropic lines of the Plücker quadric $\mathcal{Q}_P$. The focal lattices of the congruence represent asymptotic directions of the lattice.*

Remark 9. The lattices in $\mathcal{Q}_P$, given by the bi-vectors $\mathfrak{p}_1$ and $\mathfrak{p}_2$, which represent two families of asymptotic tangents of the asymptotic lattice, are Laplace transforms of each other and satisfy the following discrete Laplace equations:

$$\begin{aligned}\mathfrak{p}_{1(12)} &= \frac{P_{(2)}B_{(1)}}{PH}\mathfrak{p}_{1(1)} + \frac{A_{(2)}}{H}\mathfrak{p}_{1(2)} - \frac{P_{(2)}B_{(1)}A}{PH}\mathfrak{p}_1 \\ \mathfrak{p}_{2(12)} &= \frac{Q_{(1)}A_{(2)}}{QH}\mathfrak{p}_{2(2)} + \frac{B_{(1)}}{H}\mathfrak{p}_{2(1)} - \frac{Q_{(1)}A_{(2)}B}{QH}\mathfrak{p}_2.\end{aligned} \tag{34}$$

Finally, let us consider the line-geometric interpretation of the W congruences.

Remark 10. In contrast to the continuous case, the discrete W congruence is not a congruence in the sense of the definition used in the theory of transformations of quadrilateral lattices (see the discussion in [8]).

The bi-vector

$$\mathfrak{q} = \Theta \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix} \wedge \begin{pmatrix} \boldsymbol{x}' \\ 1 \end{pmatrix} \tag{35}$$

represents, in a convenient gauge, the line connecting $\boldsymbol{x}$ and $\boldsymbol{x}'$. From the decomposition (27) (we again take $\boldsymbol{x}'$ from formula (24) with the upper sign) and the Lelieuvre representation (12) we obtain that

$$\mathfrak{q} = x^1\mathfrak{p}_1 - x^2\mathfrak{p}_2. \tag{36}$$

The linear problems (28) and (33) and equations (17) imply that $\mathfrak{q}$ satisfies the Laplace equation

$$\mathfrak{q}_{(12)} = \frac{x^2_{(2)}}{x^2}F^2B\mathfrak{q}_{(1)} + \frac{x^1_{(1)}}{x^1}F^2A\mathfrak{q}_{(2)} - \frac{x^1_{(1)}x^2_{(2)}}{x^1x^2}F^2AB\mathfrak{q}. \tag{37}$$

Theorem 4 [8]. *Discrete W congruences are represented by two-dimensional quadrilateral lattices in the Plücker quadric $\mathcal{Q}_P$.*

Remark 11. The W congruences provide an example of quadrilateral lattices subjected to a quadratic constraint. A general theory of such quadratic reductions of quadrilateral lattices was studied in [7].

Remark 12. Since the points of intersection of the Plücker quadric with a plane represent a regulus (one family of generators of a ruled quadric in $\mathbb{P}^3$) then [8] four neighbouring lines of a W congruence are lines of the same regulus. This property of W congruences can be used to define them without using the notion of the asymptotic lattice.

3. Integrable reductions of asymptotic lattices

In this section we consider two basic integrable reductions of the asymptotic lattices: the Bianchi lattice and the Fubini–Ragazzi lattice, which are the integrable discretizations of the Bianchi and Fubini–Ragazzi surfaces respectively. These integrable reductions are obtained:

(i) imposing suitable nonlinear constraints on the geometric data of the asymptotic lattice (hence obtaining a nonlinear system of equations, characterized by the discrete Moutard equation and by the nonlinear constraint); and then:
(ii) showing that these constraints are preserved by the discrete Moutard transformation (which therefore allows one to obtain, through a sequence of linear steps only, a new solution of the above nonlinear lattice equations from a given one).

3.1. The discrete analogue of the Bianchi–Ernst system

In 1905 Bianchi introduced, as a reduction of the Moutard transformation (see appendix C), the Darboux–Bäcklund type transformation for the system

$$\begin{aligned} &\boldsymbol{N}(u,v)_{,uv} = f(u,v)\boldsymbol{N}(u,v) \\ &(\boldsymbol{N}(u,v)\cdot\boldsymbol{N}(u,v))_{,uv} = 0 \end{aligned} \tag{38}$$

where $\boldsymbol{N}(u,v)$ belongs to n-dimensional Euclidean vector space (we denote the scalar product by '$\cdot$') and a comma denotes partial differentiation with respect to the parameters that stand after it. Bianchi also introduced a permutability theorem for this system. Let us first introduce a new permutability theorem for the system (38):

Theorem 5. *Given a solution $\boldsymbol{N}$ of the continuous Bianchi–Ernst system (38) and given two transforms of it: the first one, denoted by $\boldsymbol{N}^{(1)}$ (see [1,20]), with the transformation parameter k^1, and the second one $\boldsymbol{N}^{(2)}$, with the transformation parameter k^2, satisfying besides the Moutard transformation (see appendix C)*

$$\begin{aligned} &(\boldsymbol{N}+\boldsymbol{N}^{(1)})_{,u} = \frac{\Theta^{(1)}{}_{,u}}{\Theta^{(1)}}(\boldsymbol{N}-\boldsymbol{N}^{(1)}) \qquad (\boldsymbol{N}-\boldsymbol{N}^{(1)})_{,v} = \frac{\Theta^{(1)}{}_{,v}}{\Theta^{(1)}}(\boldsymbol{N}+\boldsymbol{N}^{(1)}) \\ &(\boldsymbol{N}-\boldsymbol{N}^{(2)})_{,u} = \frac{\Theta^{(2)}{}_{,u}}{\Theta^{(2)}}(\boldsymbol{N}+\boldsymbol{N}^{(2)}) \qquad (\boldsymbol{N}+\boldsymbol{N}^{(2)})_{,v} = \frac{\Theta^{(2)}{}_{,v}}{\Theta^{(2)}}(\boldsymbol{N}-\boldsymbol{N}^{(2)}) \end{aligned} \tag{39}$$

also the following relations (see [1,20]):

$$\begin{aligned} &\boldsymbol{N}\cdot\boldsymbol{N} = \boldsymbol{N}^{(1)}\cdot\boldsymbol{N}^{(1)} = \boldsymbol{N}^{(2)}\cdot\boldsymbol{N}^{(2)} = U(u)+V(v) \\ &\boldsymbol{N}\cdot\boldsymbol{N}^{(1)} = V(v)-U(u)+2k_1 \qquad \boldsymbol{N}\cdot\boldsymbol{N}^{(2)} = U(u)-V(v)+2k_2. \end{aligned} \tag{40}$$

Then there exists the solution $\boldsymbol{N}^{(12)}$ of the continuous Bianchi–Ernst system, given in algebraic terms by

$$\boldsymbol{N}^{(12)} = -\boldsymbol{N} + \frac{4k^1+4k^2}{(\boldsymbol{N}^{(1)}+\boldsymbol{N}^{(2)})\cdot(\boldsymbol{N}^{(1)}+\boldsymbol{N}^{(2)})}(\boldsymbol{N}^{(1)}+\boldsymbol{N}^{(2)}). \tag{41}$$

Proof. We first show that $\boldsymbol{N}^{(12)}$, defined in (41), is a Moutard transform of $\boldsymbol{N}^{(1)}$ and of $\boldsymbol{N}^{(2)}$. To do that we have to check that the function η, which due to equations (C.5), (41) must be of the form

$$\eta = \frac{\Theta^1\Theta^2}{4k^1+4k^2}(\boldsymbol{N}^{(1)}+\boldsymbol{N}^{(2)})\cdot(\boldsymbol{N}^{(1)}+\boldsymbol{N}^{(2)}) \tag{42}$$

does satisfy equations (C.6). This can be verified directly using formulae (39) and (40). Now, since $\boldsymbol{N}^{(12)}$ exists and describes an (continuous) asymptotic net, (i.e. satisfies a Moutard equation (C.7)) one can easily show, using (41) and (40), that

$$\boldsymbol{N}^{(12)} \cdot \boldsymbol{N}^{(12)} = \boldsymbol{N} \cdot \boldsymbol{N}. \tag{43}$$

□

Following a standard procedure [18], we can reinterpret the superposition principle (41) for (38) as the integrable discretization (1) of the continuous Bianchi–Ernst system (38). In what follows we show how to construct solutions of the discrete Bianchi–Ernst system (1) through a sequence of linear steps only, giving a proof of the integrability of (1), alternative to that given in [28]. We restrict this proof, without loss of generality, to the three-dimensional case, equipping $\mathbb{R}^3$ with the scalar product '$\cdot$':

$$\boldsymbol{A} \cdot \boldsymbol{B} := A_0 B_0 + \epsilon(A_1 B_1 + A_2 B_2) \qquad \epsilon = \pm 1. \tag{44}$$

We remark that equation (1) is equivalent to the discrete Moutard equation (13), supplemented by the constraint

$$(\boldsymbol{N}_{(12)} + \boldsymbol{N}) \cdot (\boldsymbol{N}_{(1)} + \boldsymbol{N}_{(2)}) = U(m_1) + V(m_2). \tag{45}$$

We also assume, for simplicity, that $U(m_1) + V(m_2) > 0$.

In order to construct a suitable reduction of the Moutard transformation which would preserve the constraint (45) it is important to note the following condition.

Theorem 6. *If $\boldsymbol{N}$ and $\boldsymbol{N}'$ are connected by the discrete Moutard transformation (20), then the condition*

$$(\boldsymbol{N}_{(12)} + \boldsymbol{N}) \cdot (\boldsymbol{N}_{(1)} + \boldsymbol{N}_{(2)}) = (\boldsymbol{N}'_{(12)} + \boldsymbol{N}') \cdot (\boldsymbol{N}'_{(1)} + \boldsymbol{N}'_{(2)}) \tag{46}$$

is equivalent to the constraint (45) supplemented by equations

$$\begin{aligned} (\boldsymbol{N}'_{(1)} \mp \boldsymbol{N}) \cdot (\boldsymbol{N}' \mp \boldsymbol{N}_{(1)}) &= U(m_1) \mp k \\ (\boldsymbol{N}'_{(2)} \pm \boldsymbol{N}) \cdot (\boldsymbol{N}' \pm \boldsymbol{N}_{(2)}) &= V(m_2) \pm k \end{aligned} \tag{47}$$

where $U(m_1)$ and $V(m_2)$ are functions of single variables only and k is a constant.

Remark 13. Notice that, if we consider the transformation direction (denoted by prime) as a shift in a third variable and make use of the freedom of the form of the Moutard equation (see remark 4), then the constraint (47) on the discrete Moutard transformation is itself a discrete Bianchi–Ernst constraint (45).

The integrability of the Bianchi–Ernst lattices is the consequence of the following result [23].

Theorem 7. *Given a solution $\boldsymbol{N}$ of the Bianchi–Ernst system (13), (45) and given the ϵ-unit vectors $\boldsymbol{n}_1, \boldsymbol{n}_2$ (i.e. $\boldsymbol{n}_1 \cdot \boldsymbol{n}_1 = \boldsymbol{n}_2 \cdot \boldsymbol{n}_2 = \epsilon$) orthogonal to $\boldsymbol{N}_{(12)} + \boldsymbol{N} =: \boldsymbol{n}_0$ and to each other, then:*

(i) The linear system

$$\begin{pmatrix} \Theta \\ \Theta_{(1)} \\ \Theta_{(2)} \\ y^1 \\ y^2 \end{pmatrix}_{(1)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{Y_{(1)} \frac{p_0^0}{F} - b}{a_{(1)}} & \frac{bF - Y_{(1)}(\frac{Y+b}{Y} p_0^0 - \frac{1}{F_{(1)}})}{a_{(1)}} & \frac{b}{a_{(1)}}\left(F - \frac{Y_{(1)}}{Y} p_0^0\right) & \mp \frac{Y_{(1)}}{a_{(1)}} p_1^0 & \mp \frac{Y_{(1)}}{a_{(1)}} p_2^0 \\ -1 & F & F & 0 & 0 \\ \mp \frac{p_0^1}{F} & \pm \frac{Y+b}{Y} p_0^1 & \pm \frac{b}{Y} p_0^1 & p_1^1 & p_2^1 \\ \mp \frac{p_0^2}{F} & \pm \frac{Y+b}{Y} p_0^2 & \pm \frac{b}{Y} p_0^2 & p_1^2 & p_2^2 \end{pmatrix} \begin{pmatrix} \Theta \\ \Theta_{(1)} \\ \Theta_{(2)} \\ y^1 \\ y^2 \end{pmatrix}$$

$$\begin{pmatrix} \Theta \\ \Theta_{(1)} \\ \Theta_{(2)} \\ y^1 \\ y^2 \end{pmatrix}_{(2)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ -1 & F & F & 0 & 0 \\ \frac{Y_{(2)} \frac{q_0^0}{F} - a}{b_{(2)}} & \frac{a}{b_{(2)}}\left(F - \frac{Y_{(2)}}{Y} q_0^0\right) & \frac{aF - Y_{(2)}(\frac{Y+a}{Y} q_0^0 - \frac{1}{F_{(2)}})}{b_{(2)}} & \pm \frac{Y_{(2)}}{b_{(2)}} q_1^0 & \pm \frac{Y_{(2)}}{b_{(2)}} q_2^0 \\ \pm \frac{q_0^1}{F} & \mp \frac{a}{Y} q_0^1 & \mp \frac{a+Y}{Y} q_0^1 & q_1^1 & q_2^1 \\ \pm \frac{q_0^2}{F} & \mp \frac{a}{Y} q_0^2 & \mp \frac{a+Y}{Y} q_0^2 & q_1^2 & q_2^2 \end{pmatrix} \begin{pmatrix} \Theta \\ \Theta_{(1)} \\ \Theta_{(2)} \\ y^1 \\ y^2 \end{pmatrix} \tag{48}$$

where F, Y, a, b are given by the equations

$$F = \frac{U(m_1) + V(m_2)}{(\boldsymbol{N}_{(1)} + \boldsymbol{N}_{(2)}) \cdot (\boldsymbol{N}_{(1)} + \boldsymbol{N}_{(2)})} \tag{49}$$

$$Y = U(m_1) + V(m_2) \qquad a = U(m_1) \mp k \qquad b = V(m_2) \pm k. \tag{50}$$

and p_B^A, q_B^A, $A, B = 0, 1, 2$ are defined by the unique decompositions (we use the summation convention)

$$\boldsymbol{n}_A = p_A^B \boldsymbol{n}_{B(1)} \qquad \boldsymbol{n}_A = q_A^B \boldsymbol{n}_{B(2)} \tag{51}$$

is compatible.

(ii) The solution $(\Theta, \Theta_{(1)}, \Theta_{(2)}, y^1, y^2)$ of the system (48) satisfies the constraint

$$\epsilon[(y^1)^2 + (y^2)^2] + \frac{Y}{F}\Theta^2 + FY\left(-\frac{a}{Y}\Theta_{(1)} + \frac{b}{Y}\Theta_{(2)}\right)^2 - 2\Theta(a\Theta_{(1)} + b\Theta_{(2)}) = 0 \tag{52}$$

provided that such a constraint is satisfied at the initial point.

(iii) Given the solution $(\Theta, \Theta_{(1)}, \Theta_{(2)}, y^1, y^2)$ of the system (48) satisfying the constraint (52), then $\boldsymbol{N}'$, constructed via the equation

$$\boldsymbol{N}' = \frac{1}{2}\left(\pm \boldsymbol{N}_{(1)} \mp \boldsymbol{N}_{(2)}\right) + \frac{y^A}{2\Theta}\boldsymbol{n}_A \tag{53}$$

with y^0 given by

$$y^0 = \frac{\mp\Theta_{(1)}(U \mp k) \pm \Theta_{(2)}(V \pm k)}{U + V} \tag{54}$$

is a new solution of the discrete Bianchi–Ernst system.

Remark 14. The parameter k, present in the linear system (48), is called the transformation parameter. The linear system can also be interpreted as a nonstandard Lax pair (zero-curvature representation) of the discrete Bianchi–Ernst system, with spectral parameter k.

In our recent paper we announced the theorem on the permutability of the superposition of discrete Bianchi transformations.

Theorem 8 [23]. *Given a solution $\boldsymbol{N}$ of the Bianchi–Ernst system and given two transforms of it: the upper-sign transform $\boldsymbol{N}^{(1)}$, with the transformation parameter k^1, and the lower-sign transform $\boldsymbol{N}^{(2)}$, with the transformation parameter k^2, then there exists the unique solution $\boldsymbol{N}^{(12)}$ of the Bianchi–Ernst system, given in algebraic terms by*

$$\boldsymbol{N}^{(12)} = -\boldsymbol{N} + \frac{k^1 + k^2}{(\boldsymbol{N}^{(1)} + \boldsymbol{N}^{(2)}) \cdot (\boldsymbol{N}^{(1)} + \boldsymbol{N}^{(2)})} (\boldsymbol{N}^{(1)} + \boldsymbol{N}^{(2)}) \tag{55}$$

which is simultaneously the lower-sign transform of $\boldsymbol{N}^{(1)}$, with the transformation parameter k^2, and the upper-sign transform of $\boldsymbol{N}^{(2)}$, with the transformation parameter k^1.

Proof. We first show that $\boldsymbol{N}^{(12)}$ defined in (55) is a Moutard transform of $\boldsymbol{N}^{(1)}$ and of $\boldsymbol{N}^{(2)}$. To do that we have to check that the function Ξ, which due to equation (23) must be of the form

$$\Xi = \frac{\Theta^1\Theta^2}{k^1 + k^2} (\boldsymbol{N}^{(1)} + \boldsymbol{N}^{(2)}) \cdot (\boldsymbol{N}^{(1)} + \boldsymbol{N}^{(2)}) \tag{56}$$

does satisfy equations (22). This can be verified directly using formulae (20) and (47) applied to $\boldsymbol{N}^{(1)}$ and $\boldsymbol{N}^{(2)}$.

Now, since $\boldsymbol{N}^{(12)}$ exists and describes an asymptotic lattice, it is enough to show that the equations (47) with the correct constant and sign apply also for the pair $(\boldsymbol{N}^{(12)}, \boldsymbol{N}^{(1)})$ and for the pair $(\boldsymbol{N}^{(12)}, \boldsymbol{N}^{(2)})$, which can be done by direct calculation using (20), (45) and (55). □

Remark 15. Notice that the superposition formula (55) for the Bianchi–Ernst system reproduces the Bianchi–Ernst system itself, after replacing the upper transformation indices by the lower translation ones.

Remark 16. If we treat the transformations as shifts in additional parameters (denoted by m^1 and m^2), then the vector function $\boldsymbol{N}(m_1, m_2, m^1, m^2)$ satisfies the discrete Bianchi–Ernst system in every pair of parameters (see also remarks 5 and 13). A similar consideration in connection with the relation between the superposition formula for the discrete Tzitzeica equation and the self-dual Einstein spaces appeared in [27].

Let us introduce the Ernst form of the Bianchi–Ernst system. We propose two different versions of it. The first one comes from the observation that, if $\boldsymbol{N}$ satisfies the Bianchi–Ernst system, then function ρ given by

$$\rho := \boldsymbol{N} \cdot \boldsymbol{N}$$

satisfies

$$\left(\frac{\rho_{(12)} - \rho}{F}\right)_{(12)} - \frac{\rho_{(12)} - \rho}{F} = \left(F(\rho_{(1)} - \rho_{(2)})\right)_{(1)} - \left(F(\rho_{(1)} - \rho_{(2)})\right)_{(2)}. \tag{57}$$

After the stereographical projection

$$\begin{aligned} N_0 &= \sqrt{\rho}\frac{1 - \epsilon\xi\bar{\xi}}{1 + \epsilon\xi\bar{\xi}} \\ N_1 + \mathrm{i}N_2 &= \sqrt{\rho}\frac{2\xi}{1 + \epsilon\xi\bar{\xi}} \end{aligned} \tag{58}$$

we obtain the following:

Proposition 9. *Every solution of the discrete Bianchi–Ernst system gives rise to a solution of the system of equations*

$$
\begin{gathered}
F=\frac{\left(\sqrt{\rho}\frac{1-\epsilon\xi\bar{\xi}}{1+\epsilon\xi\bar{\xi}}\right)_{(12)}+\left(\sqrt{\rho}\frac{1-\epsilon\xi\bar{\xi}}{1+\epsilon\xi\bar{\xi}}\right)}{\left(\sqrt{\rho}\frac{1-\epsilon\xi\bar{\xi}}{1+\epsilon\xi\bar{\xi}}\right)_{(1)}+\left(\sqrt{\rho}\frac{1-\epsilon\xi\bar{\xi}}{1+\epsilon\xi\bar{\xi}}\right)_{(2)}} \\
\left(\frac{\rho_{(12)}-\rho}{F}\right)_{(12)}-\frac{\rho_{(12)}-\rho}{F}=\left(F(\rho_{(1)}-\rho_{(2)})\right)_{(1)}-\left(F(\rho_{(1)}-\rho_{(2)})\right)_{(2)} \\
\frac{\sqrt{\rho}_{(12)}(1-\epsilon\xi\bar{\xi})_{(12)}(1+\epsilon\xi\bar{\xi})+\sqrt{\rho}(1-\epsilon\xi\bar{\xi})(1+\epsilon\xi\bar{\xi})_{(12)}}{\sqrt{\rho}_{(1)}(1-\epsilon\xi\bar{\xi})_{(1)}(1+\epsilon\xi\bar{\xi})_{(2)}+\sqrt{\rho}_{(2)}(1-\epsilon\xi\bar{\xi})_{(2)}(1+\epsilon\xi\bar{\xi})_{(1)}} \\
=\frac{\sqrt{\rho}_{(12)}\xi_{(12)}(1+\epsilon\xi\bar{\xi})+\sqrt{\rho}\xi(1+\epsilon\xi\bar{\xi})_{(12)}}{\sqrt{\rho}_{(1)}\xi_{(1)}(1+\epsilon\xi\bar{\xi})_{(2)}+\sqrt{\rho}_{(2)}\xi_{(2)}(1+\epsilon\xi\bar{\xi})_{(1)}}
\end{gathered}
\tag{59}
$$

which we call the hyperbolic discrete Ernst equation.

The second proposition comes from the observation that the vector field $\boldsymbol{\nu} := \frac{\boldsymbol{N}_{(12)}+\boldsymbol{N}}{\sqrt{F}} = (\nu_0, \nu_1, \nu_2)$ is a solution of

$$\frac{\boldsymbol{\nu}_{(12)}}{\sqrt{F_{(12)}}}+\frac{\boldsymbol{\nu}}{\sqrt{F}}=\sqrt{F_{(1)}}\boldsymbol{\nu}_{(1)}+\sqrt{F_{(2)}}\boldsymbol{\nu}_{(2)} \tag{60}$$

$$\boldsymbol{\nu}\cdot\boldsymbol{\nu}=\nu_0^2+\epsilon(\nu_1^2+\nu_2^2)=U(m_1)+V(m_2)=:r \qquad \epsilon=\pm1 \tag{61}$$

provided that $\boldsymbol{N}$ is a solution of (13), (45). Introducing the stereographic change of variables

$$\nu=(\nu_0,\nu_1,\nu_2) \qquad \nu\cdot\nu=\nu_0^2+\epsilon(\nu_1^2+\nu_2^2)=r \qquad \epsilon=\pm1 \tag{62}$$

$$
\begin{gathered}
\xi=\frac{\nu_1+\mathrm{i}\nu_2}{\sqrt{r}+\nu_0} \quad\Rightarrow\quad \nu_0=\sqrt{r}\frac{1-\epsilon|\xi|^2}{1+\epsilon|\xi|^2} \qquad \nu_1=\sqrt{r}\frac{\xi+\bar{\xi}}{1+\epsilon|\xi|^2} \\
\nu_2=\sqrt{r}\frac{\xi-\bar{\xi}}{\mathrm{i}(1+\epsilon|\xi|^2)}
\end{gathered}
\tag{63}
$$

we obtain the following:

Proposition 10. *On eliminating function F from the equation (60) we get*

$$
\begin{aligned}
&\{[(1-\epsilon|\xi_{(12)}|^2)(\xi\bar{\xi}_{(2)}-\xi_{(2)}\bar{\xi})+(1-\epsilon|\xi_{(2)}|^2)(\xi_{(12)}\bar{\xi}-\xi\bar{\xi}_{(12)}) \\
&\qquad +(1-\epsilon|\xi|^2)(\xi_{(2)}\bar{\xi}_{(12)}-\xi_{(12)}\bar{\xi}_{(2)})](1+\epsilon|\xi_{(1)}|^2)^2r_{(12)}\} \\
&\qquad \times\{[(1-\epsilon|\xi_{(1)}|^2)(\xi\bar{\xi}_{(2)}-\xi_{(2)}\bar{\xi})+(1-\epsilon|\xi_{(2)}|^2)(\xi_{(1)}\bar{\xi}-\xi\bar{\xi}_{(1)})+(1-\epsilon|\xi|^2) \\
&\qquad \times(\xi_{(2)}\bar{\xi}_{(1)}-\xi_{(1)}\bar{\xi}_{2)})](1+\epsilon|\xi_{(12)}|^2)^2r_{(1)}\}^{-1} \\
&\quad =\{[(1-\epsilon|\xi_{(11)}|^2)(\xi_{(1)}\bar{\xi}_{(112)}-\xi_{(112)}\bar{\xi}_{(1)})+(1-\epsilon|\xi_{(112)}|^2)(\xi_{(11)}\bar{\xi}_{(1)}-\xi_{(1)}\bar{\xi}_{(11)}) \\
&\qquad +(1-\epsilon|\xi_{(1)}|^2)(\xi_{(112)}\bar{\xi}_{(11)}-\xi_{(11)}\bar{\xi}_{(112)})]\} \\
&\qquad \times\{[(1-\epsilon|\xi_{(11)}|^2)(\xi_{(112)}\bar{\xi}_{(12)}-\xi_{(12)}\bar{\xi}_{(112)})+(1-\epsilon|\xi_{(12)}|^2) \\
&\qquad \times(\xi_{(11)}\bar{\xi}_{(112)}-\xi_{(112)}\bar{\xi}_{(11)})+(1-\epsilon|\xi_{(112)}|^2)(\xi_{(12)}\bar{\xi}_{(11)}-\xi_{(11)}\bar{\xi}_{(12)})]\}^{-1} \qquad (64) \\
&\{[(1-\epsilon|\xi_{(1)}|^2)(\xi\bar{\xi}_{(12)}-\xi_{(12)}\bar{\xi})+(1-\epsilon|\xi_{(12)}|^2)(\xi_{(1)}\bar{\xi}-\xi\bar{\xi}_{(1)}) \\
&\qquad +(1-\epsilon|\xi|^2)(\xi_{(12)}\bar{\xi}_{(1)}-\xi_{(1)}\bar{\xi}_{(12)})](1+\epsilon|\xi_{(2)}|^2)^2r_{(12)}\} \\
&\qquad \times\{[(1-\epsilon|\xi_{(1)}|^2)(\xi\bar{\xi}_{(2)}-\xi_2\bar{\xi})+(1-\epsilon|\xi_{(2)}|^2)(\xi_{(1)}\bar{\xi}-\xi\bar{\xi}_{(1)})+(1-\epsilon|\xi|^2) \\
&\qquad \times(\xi_{(2)}\bar{\xi}_{(1)}-\xi_{(1)}\bar{\xi}_{(2)})](1+\epsilon|\xi_{(12)}|^2)^2r_{(2)}\}^{-1} \\
&\quad =\{[(1-\epsilon|\xi_{(122)}|^2)(\xi_{(2)}\bar{\xi}_{(22)}-\xi_{(22)}\bar{\xi}_{(2)})+(1-\epsilon|\xi_{(22)}|^2) \\
&\qquad \times(\xi_{(122)}\bar{\xi}_{(2)}-\xi_{(2)}\bar{\xi}_{(122)})+(1-\epsilon|\xi_{(2)}|^2)(\xi_{(22)}\bar{\xi}_{(122)}-\xi_{(122)}\bar{\xi}_{(22)})]\}
\end{aligned}
$$

$$\times\{[(1-\epsilon|\xi_{(12)}|^2)(\xi_{(122)}\bar{\xi}_{(22)}-\xi_{(22)}\bar{\xi}_{(122)})+(1-\epsilon|\xi_{(22)}|^2)$$
$$\times(\xi_{(12)}\bar{\xi}_{(122)}-\xi_{(122)}\bar{\xi}_{(12)})+(1-\epsilon|\xi_{(122)}|^2)(\xi_{(22)}\bar{\xi}_{(12)}-\xi_{(12)}\bar{\xi}_{(22)})]\}^{-1}. \tag{65}$$

This is a system of two real equations for ξ *and* $\bar{\xi}$.

3.2. The discrete analogue of the Fubini–Ragazzi system

Let us impose the symmetric reduction condition (A.3) on the equation of the discrete tangent canonical fields (10) of a asymptotic lattice, i.e.

$$\frac{\rho_{(12)}\rho}{\rho_{(1)}\rho_{(2)}}=\frac{H_{(2)}}{H_{(1)}}\qquad \rho:=\frac{\mathcal{P}}{\mathcal{Q}}. \tag{66}$$

The constraint (66) with equations (5) and (6) gives the discrete version of the Fubini–Ragazzi system [14,21]. The Darboux–Bäcklund transformation for the discrete Fubini–Ragazzi comes directly from the discrete Jonas formulae of section 2.5 (details can be found in [22]).

Theorem 11. *Consider a Fubini–Ragazzi lattice* $\boldsymbol{x}$ *with the normal* $\boldsymbol{N}$, *and let the function* ζ *be the solution of the system*

$$\begin{aligned}\frac{\zeta_{(2)}}{\zeta}&=\frac{H(QF^2)_{(12)}}{Q_{(1)}}\\ \frac{\zeta_{(1)}}{\zeta}&=\frac{H(PF^2)_{(12)}}{P_{(2)}}\end{aligned} \tag{67}$$

(i.e. ζ *is given up to a constant parameter, say k) then:*

(i) The linear system

$$\begin{pmatrix}x^1\\x^2\\x^3\\\Theta_{(1)}\\\Theta_{(2)}\\\Theta\end{pmatrix}_{(1)}=\begin{pmatrix}0 & -\frac{1}{FA} & -\frac{1}{A} & -\frac{1}{A} & 0 & 0\\ 0 & \frac{A-P}{FA} & -\frac{P}{A} & -\frac{P}{A} & 0 & 0\\ 1 & \frac{C}{AFF_{(1)}}-1 & \frac{C}{AF_{(1)}} & \frac{C}{AF_{(1)}} & 0 & -1\\ 0 & \frac{-\zeta}{FAQ_{(1)}(F_{(1)})^2} & \frac{-\zeta}{AQ_{(1)}(F_{(1)})^2} & \frac{C-PFF_{(1)}}{F_{(1)}}-\frac{\zeta}{AQ_{(1)}(F_{(1)})^2} & -PF & P-A\\ 0 & 0 & 0 & F & F & -1\\ 0 & 0 & 0 & 1 & 0 & 0\end{pmatrix}\begin{pmatrix}x^1\\x^2\\x^3\\\Theta_{(1)}\\\Theta_{(2)}\\\Theta\end{pmatrix}$$
$$\begin{pmatrix}x^1\\x^2\\x^3\\\Theta_{(1)}\\\Theta_{(2)}\\\Theta\end{pmatrix}_{(2)}=\begin{pmatrix}\frac{B-Q}{FB} & 0 & -\frac{Q}{B} & 0 & \frac{Q}{B} & 0\\ -\frac{1}{FB} & 0 & -\frac{1}{B} & 0 & \frac{1}{B} & 0\\ \frac{D}{BFF_{(2)}}-1 & 1 & \frac{D}{BF_{(2)}} & 0 & -\frac{D}{BF_{(2)}} & 1\\ 0 & 0 & 0 & F & F & -1\\ \frac{-\zeta}{FBP_{(2)}(F_{(2)})^2} & 0 & \frac{-\zeta}{BP_{(2)}(F_{(2)})^2} & -QF & \frac{D-QFF_{(2)}}{F_{(2)}}+\frac{\zeta}{BP_{(2)}(F_{(2)})^2} & Q-B\\ 0 & 0 & 0 & 0 & 1 & 0\end{pmatrix}\begin{pmatrix}x^1\\x^2\\x^3\\\Theta_{(1)}\\\Theta_{(2)}\\\Theta\end{pmatrix} \tag{68}$$

is compatible.

(ii) Given the solution $(x^1, x^2, x^3, \Theta_{(1)}, \Theta_{(2)}, \Theta)$ *of the system (68) then* $\boldsymbol{x}'$, *constructed via the upper-sign formula (24), with* $\boldsymbol{N}'$ *constructed via equation (27), is a new Fubini–Ragazzi lattice. The corresponding new solution of equations (5), (6), subject to the constraint (66), is given by formulae (30), (31).*

Remark 17. The parameter k, present in the linear system (68) via equations (67), is called the transformation parameter. The linear system can also be interpreted as the Lax pair (zero-curvature representation) of the discrete Fubini–Ragazzi system, with spectral parameter k.

Remark 18. The discrete analogue of (indefinite) affine spheres [3, 4], which is described by the discrete analogue of the Tzitzeica equation, is the particular reduction of the discrete Fubini–Ragazzi lattice corresponding to

$$\begin{aligned} C &= F_{(1)}(1 + A + P) \\ D &= F_{(2)}(1 + B + Q) \end{aligned} \qquad (69)$$

i.e. $\gamma = \delta = 0$ in equations (16).

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Appendix A. Quadrilateral lattices

We will need a few basic facts from the theory of quadrilateral lattices, which are the discrete integrable analogues of conjugate nets [6,9,26]. The N-dimensional quadrilateral lattice in $\mathbb{P}^M$, $2 \leqslant N \leqslant M$, is geometrically characterized by the planarity of the elementary quadrilaterals of the lattice. In terms of the homogeneous representation $\boldsymbol{y} : \mathbb{Z}^N \to \mathbb{R}^{M+1}_*$ of the lattice, this geometric characterization can be algebraically expressed as a linear constraint between $\boldsymbol{y}$, $\boldsymbol{y}_{(i)}$, $\boldsymbol{y}_{(j)}$ and $\boldsymbol{y}_{(ij)}$, where $i \neq j$ and $i, j = 1, \ldots, N$. In the generic case, such a linear relation can be put in the form of the so-called discrete Laplace equation [6, 9]

$$\boldsymbol{y}_{(ij)} = \mathcal{A}_{ij}\boldsymbol{y}_{(i)} + \mathcal{A}_{ji}\boldsymbol{y}_{(j)} + \mathcal{C}_{ij}\boldsymbol{y} \qquad i \neq j \quad \mathcal{C}_{ij} = \mathcal{C}_{ji}. \qquad \text{(A.1)}$$

From the theory of the Darboux-type transformations of the quadrilateral lattices [11] we recall that a $\mathbb{Z}^N$-parameter family of lines such that any two neighbouring lines intersect is called a (N-dimensional) discrete congruence. The intersection points of lines of the congruence with their i-direction neighbours define the ith focal lattice of the congruence. Such focal lattices are, in general, quadrilateral lattices. Any two focal lattices of the congruence are connected by the so-called Laplace transformation [6].

In the affine gauge the system of Laplace equations (A.1) can be replaced by the following linear system: (see also [5])

$$\Delta_j \boldsymbol{X}_i = Q_{ij(j)}\boldsymbol{X}_j \qquad i \neq j. \qquad \text{(A.2)}$$

An important integrable reduction of the quadrilateral lattice is the so-called symmetric reduction [10]. Among various characterizations of the symmetric lattice we will use the following constraint:

$$\frac{r_{ij(ij)}r_{ij}}{r_{ij(i)}r_{ij(j)}} = \frac{(1 - Q_{ji(i)}Q_{ij(j)})_{(i)}}{(1 - Q_{ji(i)}Q_{ij(j)})_{(j)}} \qquad i \neq j \qquad \text{(A.3)}$$

where

$$r_{ij} := \frac{Q_{ij(j)}}{Q_{ji(i)}} \qquad i \neq j. \qquad \text{(A.4)}$$

Appendix B. The line geometry of Plücker

In the line geometry the primary elements are straight lines in $\mathbb{R}^3$. It is convenient to consider $\mathbb{R}^3$ as the affine part of the projective space $\mathbb{P}^3$ (by the standard embedding $\boldsymbol{x} \mapsto [(\boldsymbol{x}, 1)^T]$) and study straight lines in that space.

The line passing through two points $[\boldsymbol{u}]$, $[\boldsymbol{v}]$ of $\mathbb{P}^3$, can be represented, up to a proportionality factor, by a bi-vector

$$\mathfrak{p} = \boldsymbol{u} \wedge \boldsymbol{v} \in \bigwedge^2(\mathbb{R}^4). \tag{B.1}$$

The space of straight lines in $\mathbb{P}^3$ can therefore be identified with a subset of $\mathbb{P}(\bigwedge^2(\mathbb{R}^4)) \simeq \mathbb{P}^5$. The necessary and sufficient condition for a non-zero bi-vector $\mathfrak{p}$ in order to represent a straight line is given by the homogeneous equation

$$\mathfrak{p} \wedge \mathfrak{p} = 0. \tag{B.2}$$

If $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_4$ is a basis of $\mathbb{R}^4$ then the following bi-vectors:

$$\boldsymbol{e}_{i_1 i_2} = \boldsymbol{e}_{i_1} \wedge \boldsymbol{e}_{i_2} \qquad 1 \leqslant i_1 < i_2 \leqslant 4 \tag{B.3}$$

form the corresponding basis of $\bigwedge^2(\mathbb{R}^4)$:

$$\mathfrak{p} = p^{12}\boldsymbol{e}_{12} + p^{13}\boldsymbol{e}_{13} + \cdots + p^{34}\boldsymbol{e}_{34}. \tag{B.4}$$

Equation (B.2) rewritten in the Plücker coordinates p^{ij} reads

$$p^{12}p^{34} - p^{13}p^{24} + p^{14}p^{23} = 0 \tag{B.5}$$

and defines in $\mathbb{P}^5$ the so-called Plücker quadric $\mathcal{Q}_P$.

Appendix C. Superposition of Moutard transformations

The map $N(u, v) \to N'(u, v)$ called the Moutard transformation

$$(N'\Theta)_{,u} = N\Theta_{,u} - N_{,u}\,\Theta \qquad (N'\Theta)_{,v} = -N\Theta_{,v} + N_{,v}\,\Theta \tag{C.1}$$

is an invertible map between solution spaces of the Moutard equations

$$N_{,uv} = fN \tag{C.2}$$

$$N'_{,uv} = f'N' \tag{C.3}$$

where functions F and F' are given by

$$f = \frac{\Theta_{,uv}}{\Theta} \qquad f' = \frac{\left(\frac{1}{\Theta}\right)_{,uv}}{\frac{1}{\Theta}}. \tag{C.4}$$

A classical permutability theorem can be presented as follows (see e.g. [13, 21]).

Theorem 12. *Given a solution N of the Moutard equation (C.2) and its two Moutard transforms: the first one denoted by $N^{(1)}$ (superscript instead of prime N' in formulae (C.1)!) governed by (C.1) with function $\Theta^{(1)}$ instead of function Θ and the second one denoted by $N^{(2)}$ governed by (C.1) with mutually interchanged parameters $u \leftrightarrow v$, with function $\Theta^{(2)}$ instead of function Θ, are given, then function $N^{(12)}$ given by*

$$N^{(12)} = -N + \frac{\Theta^{(1)}\Theta^{(2)}}{\eta}(N^{(2)} + N^{(1)}) \tag{C.5}$$

where η is given by the quadratures

$$\begin{aligned} \eta_{,u} &= \Theta^{(2)}\Theta^{(1)}{}_{,u} - \Theta^{(1)}\Theta^{(2)}{}_{,u} \\ \eta_{,v} &= -\Theta^{(2)}\Theta^{(1)}{}_{,v} + \Theta^{(1)}\Theta^{(2)}{}_{,v} \end{aligned} \tag{C.6}$$

is a solution of the Moutard equation

$$N^{(12)}{}_{,uv} = F^{(12)}N^{(12)} \tag{C.7}$$

where

$$F^{(12)} = F + \eta\left(\frac{1}{\eta}\right)_{,uv} + \frac{1}{\eta}(\Theta^{(2)}{}_{,u}\,\Theta^{(1)}{}_{,v} - \Theta^{(1)}{}_{,u}\,\Theta^{(2)}{}_{,v}).$$

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